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SAVAGE'S VISION IN DECISION-MAKING, MODELS AND ALGORITHMS

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Abstract: In the context of solving applied economic problems, uncertainty generally generates a multitude of decision-making difficulties. These require specification based on the type of the decision-maker, the environment, available resources and decision constraints as well as other factors. This article looks at one specific sequence from this multitude - namely - the one that refers to the regret criterion, known in the literature as Savage's criterion. It is important to clarify from the beginning that the number of states of nature is considered finite, the admissible decision domain is a compact and convex set in the corresponding space, and the functions generated by all states of nature - which in this case represent costs - are considered convex and continuous over the given decision domain. Evaluating the regret for each state of nature is, in itself, a difficult problem. Estimating the values of the Savage function for any specific decision option is even more challenging. This paper proposes a method for minimizing Savage's regret function, which is developed based on the well-known generalized gradient projection method. This proposed method can be implemented in multiple versions depending on the number of states of nature, the number of constraints that define the admissible decision domain as well as other factors and limitations. All these versions are implemented through specific algorithms. The convergence conditions of the algorithms are established and the application areas for these numerical schemes are indicated. The identification of sectors of the economy where the developed algorithms could be tested from various perspectives represents a particular interest in the context of this research.
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Introduction

In decision-making, the absence of regret is practically impossible. Whenever someone expresses something *regretfully* or remarks that the outcome could have been better in a given situation, it indicates the presence of regrets associated with certain decisions. In other words, alternative choices might have been possible which, under specific circumstances, could have produced more favourable outcomes than those actually obtained. Based on this concept, the present study adopts a quantitative approach to the problem of minimizing regrets under conditions of uncertainty. The concept of regret in decision-making under uncertainty, particularly for special cases, was first introduced by Savage [1]. In works [2] and [3], two decision-making problems are examined in detail, for which five algorithms have been developed. These algorithms are specifically designed to minimize the maximum regret value of Savage's regret functions.

Models description

The considered models have the following form:

Model I:

$$R_s(u) \to \min_{u \in U \subset E^n} \tag{1}$$

Model II:

$$R_s(u) \to \min_{u \in D = \{u \in U : F(u) \le 0\}} \tag{2}$$

Here:

$R_s(u) = \max_{i \in I}\{\bar{r}(u,\ q_i)\}$ - the Savage function;

$\bar{r}(u,\ q_i) = r(u,\ q_i) - r_i^*$ - the Regret function;

determined by the state of nature $q_i \in Q = \{q_1, \dots, q_m\}$;

$r_i^* = r\left(u_{(i)}^*, q_i\right) = \min r(u, q_i)$ and $u \in U$ in model I, $u \in D$ in model II;

function

$$F(u) = \max_{j \in J}\{F_j(u)\}.$$

It is assumed that all functions $r(u,\ q_i), i \in I = \{1, \dots, m\}$ and $F_j(u), j \in J = \{1, \dots, t\}$ are continuous and convex on the compact and convex U in the Euclidian space E^n.

Remark. In order to be able to work with the function $\bar{r}(u,\ q_i)$ it is required to know its minimum value r_i^* which is realistically impossible. When implementing the algorithms, in each iteration, certain estimations of r_i^*, will be used instead of its actual value.

In the following we will consider certain partitions of the sets I and J into subsets $I_1, \dots, I_M$ and $J_1, \dots, J_N$, respectively, such that:

$$\bigcup_{i=1}^{M} I_i = I, \qquad \bigcup_{j=1}^{N} J_j = J,$$

$$I_{i_1} \cap I_{i_2} = \emptyset, i_1 \ne i_2, \qquad J_{j_1} \cap J_{j_2} = \emptyset, j_1 \ne j_2,$$

$$I_i \ne \emptyset, i = \overline{1, M}. \qquad J_j \ne \emptyset, j = \overline{1, N}.$$

It will be additionally assumed that the sub gradients of the functions $r(u,\ q_i),\ i\ \in\ I$, also exist for all border points *U*, which, at the same time, are uniformly bounded in *U*. There exists a constant C_1, for which $\|r_u'(u, q_i)\| \le C_1$ for all $u \in U$ and all $i \in I$, $r_u'(u; q_i)$ being any subgradient of the function $r_u'(u; q_i)$, which corresponds to point *u*.

We define the set *U* as relatively simple, in the sense that for any point $z \in E^n$, the projection of *z* onto *U* can be determined exactly. This projection will be denoted $\Pi_U(z)$.

We will consider *m+1* iterative processes, which can run in parallel. The first *m* processes will be called internal and will determine approximations of the values r_i^* , $i \in I$; and the *(m+1)*th one will be called an external process, oriented to provide approximations of the value

$$R_S^* = \min_{u \in U} R_S\,(u)\,. \tag{3}$$

To solve problem (1) the following algorithm is proposed:

Algorithm 1. The *m+1* parallel (optimization) processes are described as follows:

$$\begin{cases} u_{(i)}^{k+1} = \Pi_U\left(u_{(i)}^k - h_{(i)k} \cdot \eta_{(i)}^k\right), i \in I \\ u^{k+1} = \Pi_U(u^k - h_k \cdot \eta^k). \end{cases} \quad (4)$$

Here:

$$\eta_{(i)}^k = \begin{cases} r'_u\left(u_{(i)}^k, q_i\right) / \left\| r'_u\left(u_{(i)}^k, q_i\right) \right\|, \text{if } r'_u\left(u_{(i)}^k, q_i\right) \neq 0 \\ 0, \text{otherwise} \end{cases} \quad (5)$$

$r'_u\left(u_{(i)}^k, q_i\right)$ - an arbitrary subgradient of the function $r(u, q_i)$ with respect to u, for $u = u_{(i)}^k$

$$\eta^k = \begin{cases} \left(R_S^k(u^k)\right)'_u / \left\| \left(R_S^k(u^k)\right)'_u \right\|, \text{if } \left(R_S^k(u^k)\right)'_u \neq 0 \\ 0, \text{otherwise} \end{cases} \quad (6)$$

where $\left(R_S^k(u^k)\right)'_u$ represents an arbitrary subgradient of the function

$$R_S^k(u^k) = \max_{i \in I}\left[r(u, q_i) - r\left(u_{(i)}^k, q_i\right)\right] \quad (7)$$

calculated in the point $u=u^k$.

Numerical strings $h_{(i)k}, h_k$ will respect the conditions:

$$h_{(i)k}, h_k \geq 0 \; ; \; h_{(i)k}, h_k \rightarrow 0 \; ; \; \sum_{k=0}^{\infty} h_{(i)k} \, , \sum_{k=0}^{\infty} h_k = \infty \quad (8)$$

Let U^* - the set of minimum points of the Savage function on the domain U. In compliance with the conditions listed above, the following statement occurs.

Theorem 1. In conditions (4)-(8), the following relations occur

$$\lim_{k \to \infty} \min_{u^* \in U^*} \|u^k - u^*\| = 0 \, , \lim_{k \to \infty} R_S^k(u^k) = R_S^* \quad (*)$$

Where R_S^* is defined in (3).

Algorithm 2. For solving problem (1).

Suppose that at iteration k the points $u_{(i)}^k, i = \overline{1,m}$ and u^k are already known. In order to determine the following set of points $u_{(i)}^{k+1}, i = \overline{1,m}$ and u^{k+1} the following steps are performed:

$$u_{(i)}^{k+1} = \begin{cases} P_U(u_{(i)}^k - h_{(i)k} \cdot \eta_{(i)}^k), \;\; \text{if } i \in \tilde{I}_{i_k} \\ u_{(i)}^k, \;\; \text{if } i \notin \tilde{I}_{i_k}, k = 0,1, \ldots \end{cases} \quad (9)$$

$$u^{k+1} = P_U(u^k - h_k \cdot \eta^k), k = 0,1, \ldots \quad (10)$$

where:

$i_k = k - M \cdot \left[\frac{k}{M}\right] + 1$, (obviously, the index i_k takes the consecutive values $1,2, \ldots, M$);

$\left[\frac{k}{M}\right]$ – the integer part of the ratio $k/_M$;

$\tilde{I}_{i_k} = I_{i_k} \cup \{i^{k-1}\}$, ($i^{-1}$ any element from I);

$$\eta_{(i)}^k = \begin{cases} gr(u_{(i)}^k, q_i) / \left\| gr(u_{(i)}^k, q_i) \right\|, \;\; \text{if } gr(u_{(i)}^k, q_i) \neq 0 \\ 0, \;\; \text{otherwise} \end{cases} \quad (11)$$

where: $gr(u_{(i)}^k, q_i)$ – is an arbitrary subgradient of the function $r(u_{(i)}, q_i)$, computed at the point $u_{(i)} = u_{(i)}^k$, and $\left\| gr(u_{(i)}^k, q_i) \right\|$ denoted the magnitude (the Euclidian norm) of this subgradient.

The vector η^k from (10) is determined in the following way:

$$\eta^k = \begin{cases} g\tilde{R}^k(u^k) / \left\| g\tilde{R}^k(u^k) \right\|, \;\; \text{for } g\tilde{R}^k(u^k) \neq 0 \\ 0, \text{otherwise} \end{cases} \quad (12)$$

$g\tilde{R}^k(u^k)$ – an arbitrary subgradient of the function:

$$\tilde{R}^k(u) = \max_{i \in \tilde{I}_{i_k}} \left\{ r(u, \; q_i) - \; r\left(u_{(i)}^k, q_i\right) \right\}, \quad (13)$$

computed at the point $u = u^k$. The value i^k represents the index i from $\tilde{I}_{i_k}$ for which the value $\tilde{R}^k(u^k)$ is attained (if there are multiple such indices, one is chosen arbitrarily).

Remark. Numbers i^k and i_k are not to be confused. Numbers $h_{(i)k}$ and h_k represent the step sizes for minimizing the functions $r(u,\ q_i)$ and $R_s(u)$ respectively, at iteration k.
It will be assumed henceforth that the sequences $h_{(i)k}, i = \overline{1,m}$ and h_k satisfy the conditions:

$$h_{(i)k}, h_k \geq 0;\ h_{(i)k}, h_k \to 0; \qquad (14)$$
$$\sum_{k=0}^{\infty} h_{(i)k\cdot M+l} = \infty, l = 0,1,\dots, M-1; \sum_{k=0}^{\infty} h_k = \infty$$

Remark. For the M relations in (14) (for $l = \overline{0, M-1}$) to hold, it is necessary that all the subsequences $\{h_{(i)k\cdot M+l}\}_{k=0}^{\infty}$, for $l = 0,1,\dots, M-1$ form divergent series.
Constructing such sequences is not difficult. For example, if a sequence $\{\tilde{h}_{(i)l}\}$ satisfies the conditions:

$$\tilde{h}_{(i)l} \geq 0, \tilde{h}_{(i)l} \to 0, \sum_{l=0}^{\infty} \tilde{h}_{(i)l} = \infty$$

and the sequence $\{h_{(i)k}\}$ is to be defined in the following way:

$$h_{(i)k} = \tilde{h}_{(i)l}, \text{ for } l \cdot M \leq k \leq (l+1)M - 1,$$

then the sequence $\{h_{(i)k}\}$ has all of its subsequences in the form $\{h_{(i)k\cdot M+l}\}$, $l = 0,1,\dots, M-1$, with the same properties as the sequence $\{\tilde{h}_{(i)l}\}$.
Theorem 2. Let U^* be the set of minimum points of problem (1). If requirements (9) - (14) are satisfied, the following statements hold:

$$\lim_{k\to\infty} \min_{u^*\in U^*} \|u^k - u^*\| = 0\ , \lim_{k\to\infty} \tilde{R}^k(u^k) = R_S(u^*). \qquad (**)$$

To solve problem (2), the third algorithm will be considered:
Algorithm 3. Assuming the existence of the solution to the problem (2), and thus to each of the *m* internal problems, the "lenient" variant will be investigated, in which the fulfillment of the restriction will be required

$$F(u) \leq \bar{\varepsilon},\ \bar{\varepsilon} > 0 \qquad (15)$$

The quantity $\bar{\varepsilon}$ will be called threshold or tolerance level.
In this case, instead of problem (2), the problem will be solved in the form (1), (15). The solutions of these problems could be considered approximate solutions of the model (2).
Remark. Internal issues are resolved with the same "threshold of tolerance" although each of them might have its own threshold of tolerance.
The numerical algorithm includes all actions (4)-(8) with the following specification for (5) and (6):

$$\eta_{(i)}^k = \begin{cases} r_u'(u_{(i)}^k, q_i)/\left\|r_u'(u_{(i)}^k, q_i)\right\|, \text{ if } F\left(u_{(i)}^k\right) \leq \bar{\varepsilon} \text{ și } r_u'(u_{(i)}^k, q_i) \neq 0 \\ g(u_{(i)}^k)/\left\|g(u_{(i)}^k)\right\|, \text{if } F(u_{(i)}^k) > \bar{\varepsilon} \text{ and } g\left(u_{(i)}^k\right) \neq 0 \\ 0, \text{in the rest of the cases. Here } i \in I \end{cases} \qquad (16)$$

$$\eta^k = \begin{cases} g_s^k(u^k)/\|g_s^k(u^k)\|, \text{if } F(u^k) \leq \bar{\varepsilon} \text{ and } g_s^k(u^k) \neq 0, \\ g(u^k)/\|g(u^k)\|, \text{if } F(u^k) > \bar{\varepsilon} \text{ and } g(u^k) \neq 0, \\ 0, \text{in the rest of the cases.} \end{cases} \qquad (17)$$

It will be admitted that the subgradients $g_s^k(u)$ and $g(u)$ of the functions $R_s^k(u)$ and $F(u)$, correspondingly, are uniformly bounded on the set U of the same constant C.
In the conditions listed, it has their affirmation:
Theorem 3. For any element $u^0 \in U$ and any number $\varepsilon > 0$, there is such a number $\bar{\varepsilon} > 0$, so that, except for a finite number, all terms of the series $\{u^k\}$, generated by the algorithm (4), (8), (16), (17), are contained in the set $V(U^*, \varepsilon)$, where U^* is the totality of the solutions to the problem (2); $V(U^*, \varepsilon)$ - the ε - neighborhood of the set U^*.
Remark. The algorithm stops if, at some iteration k, the vector $\eta^k = 0$. In this situation two cases would be possible:

1. In point u^k the inequality $F(u^k) \leq \bar{\varepsilon}$ occurs and $g_s^k(u^k) = 0$; thus u^k is one of the optimal solutions of problem (1), (15);
2. In point u^k the inequality $F(u^k) > \bar{\varepsilon}$ occurs and $g(u^k) = 0$, which means that the point u^k is the minimum point for the function $F(u)$ on the domain U and therefore the problem (1), (15) has no admissible solutions.

Algorithm 4. In algorithm 4, similar to algorithm 3, $F(u^k)$ is compared with $\bar{\varepsilon} = \bar{\varepsilon}_k$.
Takes place ***Theorem 4***. Let $h_{(i)k}, h_k$ și $\bar{\varepsilon}_k$ meet the requirements:

$$h_{(i)k}, h_k \geq 0;\ h_{(i)k}, h_k \to 0;\ \bar{\varepsilon}_k > 0; \bar{\varepsilon}_k \to 0;\ h_{(i)k}/\bar{\varepsilon}_k;$$
$$h_k/\bar{\varepsilon}_k \to 0,\ \sum_{k=0}^{\infty} h_{(i)k}\, \bar{\varepsilon}_k = \infty, \sum_{k=0}^{\infty} h_k \bar{\varepsilon}_k = \infty. \qquad (18)$$

If problem (2) has a solution, then, under the conditions of compliance with (16), (17), (18), statements (*) hold. U^* is the solution set of the problem (2).

Description of Algorithm 5. Solving Problem (2).

As in the application of Algorithm 2, the transition rules (9) and (10) are used to move from iteration (k) to the following iteration $(k+1)$. However, in this case, the method for determining the vectors $\eta_{(i)}^k$ and η^k differs. Specifically:

$$\eta_{(i)}^k = \begin{cases} gr(u_{(i)}^k, q_i)/\|gr(u_{(i)}^k, q_i)\|, \text{if } \tilde{F}_{(i)}^k(u_{(i)}^k) \leq \varepsilon_{(i)k} \text{ and } gr(u_{(i)}^k, q_i) \neq 0 \\ g\tilde{F}_{(i)}^k(u_{(i)}^k)/\|g\tilde{F}_{(i)}^k(u_{(i)}^k)\|, \text{if } \tilde{F}_{(i)}^k(u_{(i)}^k) > \varepsilon_{(i)k} \text{ and } g\tilde{F}_{(i)}^k(u_{(i)}^k) \neq 0 \\ 0, \text{otherwise.} \end{cases} \qquad (19)$$

Here:

$$gr(u_{(i)}^k, q_i) = r_u'(u_{(i)}^k, q_i)$$
$$i \in \tilde{I}_{i_k};$$
$$\tilde{F}_{(i)}^k(u) = \max_j\{F_j(u)\}, j \in \tilde{J}_{j_k}^i; \qquad (20)$$
$$\tilde{J}_{j_k}^i = \tilde{J}_{j_k} \cup \{j_i^{k-1}\};$$

j_i^{-1} – is any element from set J, $i = \overline{1, m}$;
$\varepsilon_{(i)k}$ – is the ceiling (tolerance threshold) with respect to the state of nature q_i, $i \in \tilde{I}_{i_k}$ at iteration k.

$$j_k = k - N \cdot \left[\frac{k}{N}\right] + 1, k = 0,1, \dots$$
$$\eta^k = \begin{cases} g\tilde{R}^k(u^k)/\|g\tilde{R}^k(u^k)\|, \text{if } \tilde{F}^k(u^k) \leq \varepsilon^k \text{ and } g\tilde{R}^k(u^k) \neq 0 \\ g\tilde{F}^k(u^k)/\|g\tilde{F}^k(u^k)\|, \text{if } \tilde{F}^k(u^k) > \varepsilon^k \text{ and } g\tilde{F}^k(u^k) \neq 0 \\ 0, \text{otherwise.} \end{cases} \qquad (21)$$

In (21) vectors $g\tilde{R}^k(u)$ and $g\tilde{F}^k(u)$ are arbitrary subgradients of the function $\tilde{R}^k(u)$ and $\tilde{F}^k(u)$ respectively, computed at point $u = u^k$. $\tilde{R}^k(u)$ is defined in (13), and $\tilde{F}^k(u) = \max_j\{F_j(u)\}$ with respect to all values $j \in \tilde{J}_{j_k} = \tilde{J}_{j_k} \cup \{j^{k-1}\}$; j^{-1} – being an arbitrary element from the set J.

Next, it is necessary to identify the requirements regarding the parameters $h_{(i)k}$, h_k, $\varepsilon_{(i)k}$, ε_kThe following conditions will be assumed:

$$h_{(i)k}, h_k \geq 0;\ \varepsilon_{(i)k}, \varepsilon_k > 0;\ h_{(i)k}, h_k, \varepsilon_{(i)k}, \varepsilon_k \to 0;$$
$$h_{(i)k}/\varepsilon_{(i)k}, h_k/\varepsilon_k \to 0; \qquad (22)$$
$$\sum_{k=0}^{\infty} h_{(i)k\cdot M+l} \cdot \varepsilon_{(i)k\cdot M+l} = \infty, l = 0,1, \dots, \mathrm{M}-1$$
$$\sum_{k=0}^{\infty} h_k \cdot \varepsilon_k = \infty. \qquad (23)$$

The following statement holds:

Theorem 5. Let U^* be the set of minimum points of problem (2). If the algorithm defined by rules (9) and (10) is applied, with the additional conditions (19)–(23) imposed, then, with respect to problem (2), the equalities (**) hold.

Remark. At the first M iterations ($k = \overline{0, M-1}$) index i (which represents the states of nature) goes through all the values from $I_1 \cup \{i^{k-1}\}$ (for $k = 0$), from $I_2 \cup \{i^{k-1}\}$ (for $k = 1$) and so on, from $I_M \cup \{i^{k-1}\}$ (for $k = M - 1$), and index j (which is associated with restriction j or with the function $F_j(u)$) goes through all the values from the set $J_1 \cup \{j_i^{k-1}\}$ consecutively to determine $\eta_{(i)}^k$ from (19) ($i = \overline{1, m}$) and goes through all the values from the set $J_1 \cup \{j^{k-1}\}$, to determine η^k from (21).

For the following M iterations ($k = \overline{M, 2M-1}$) i takes consecutive values from the set $I_1 \cup \{i^{k-1}\}$ (for $k = M$), ..., from $I_M \cup \{i^{k-1}\}$ (for $k = 2M - 1$), and index j from $J_2 \cup \{j_i^{k-1}\}$ - to define (19) and, respectively, from $J_2 \cup \{j^{k-1}\}$ - to define (21). And so on, for iterations $k = \overline{(N-1) \cdot M, N \cdot M - 1}$, similarly, index i, takes values from the set $I_1 \cup \{i^{k-1}\}$, $I_2 \cup \{i^{k-1}\}$,..., $I_M \cup \{i^{k-1}\}$, whereas index j – values from $J_N \cup \{j_i^{k-1}\}$, in order to determine the vector $\eta_{(i)}^k$ from (10) and values from $J_1 \cup \{j^{k-1}\}$ to determine vector η^k from (21). After $N \cdot M$ iterations, everything repeats except the numbers i^{k-1}, j_i^{k-1} and j^k can vary from iteration to iteration.

Conclusion

The theorems and algorithms presented in this paper were designed in order to minimise the values of the Savage function. This is a special function as its values and the values of its sub gradients cannot be computed. In spite of such difficult conditions, the proposed algorithms can estimate both the values of the Savage function and its subgradients to any desired level of precision. Given that certain conditions regarding the step size adjustments are met, and in more general cases, that tolerance threshold values for constraint violations are applied it is possible to ensure the convergence of the algorithms toward the set of optimal decision alternatives.

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